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# Vector-Based 3D Graphic Statics (Part I): Evaluation of Global Equilibrium

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## Abstract
The evaluation of the equilibrium of a system of forces that fulfils specified boundary conditions is a core question of theory of structures. This paper reviews three methods, related to procedures introduced at the end of the 19th century, to evaluate the global equilibrium in three dimensions using graphic statics. The paper is specifically focused on one of these methods, which is grounded on the use of projections. Based on this method, a given system of forces can be reduced to three skew resultants, which are parallel to three initially chosen unit vectors. The three resultants can be composed into two resultants thanks to the construction of a simple 3D auxiliary structure or reduced to one resultant and a couple. Given the three resultants, the reactions at the supports can be evaluated according to specified boundary conditions in both cases of externally statically determinate and indeterminate systems.

**Keywords**: vector-based 3D graphic statics, global equilibrium, funicular structure, projections

## 1. Introduction
It is a well-known property in theory of structures (Marti [5], p. 44) that a system of $n$ applied forces Σ can be associated to an equivalent force-couple system $\{\mathbf{r}, \mathbf{m_O}\}$. This consists of a pair of free vectors (vectors that are not bounded to any specific line of action), namely the resultant force $\mathbf{r}$ and the resultant couple (resultant moment) $\mathbf{m_O}$, being the latter evaluated in relation to an arbitrary reference point O.

The resultant force $\mathbf{r}$ is the result of the vectorial sum of the individual forces $\mathbf{f_i}$ constituting the system of forces:

$$\mathbf{r} = \sum_{i=0}^{n} \mathbf{f_i}$$

The resultant couple $\mathbf{m_O}$ with respect to an arbitrary reference point $O$ is given by:

$$\mathbf{m_O} = \sum_{i=0}^{n} \mathbf{p_i} \times \mathbf{f_i}$$

where $\mathbf{p_i}$ represents the position vector of a point on the line of action of the force $\mathbf{f_i}$ in relation to the chosen reference point $O$.

Two systems of forces Σ and Σ' are equivalent if their force-couple systems are identical with respect to the same arbitrary reference point.

Moreover, a system of forces is in equilibrium if its force-couple system is null:

$$\mathbf{r} = \mathbf{0} \quad , \quad \mathbf{m_O} = \mathbf{0}$$

In general, the evaluation of the global equilibrium of a given system of external forces according to specified boundary conditions can be solved following a two-phase process. At first, a force-couple system that is equivalent to the input system of forces is found. Then, a system of reaction forces based on the specified boundary conditions are obtained so that they are in equilibrium with the force-couple system. Based on the previously introduced properties, the system of reaction forces and the input system of forces constitute themselves a system of forces in equilibrium.

The evaluation of the global equilibrium is at the basis of 2D graphic statics. It involves the assessment of the magnitude of the resultant force of a given system of forces and the determination of its line of action so that the moment generated by the resultant force is equivalent to the resultant couple of the given system of forces, with respect to any arbitrary reference point in plane. In this way, the input system of forces is equivalent to a resultant force alone, with its associated line of action. The most common procedure to solve this problem within 2D graphic statics consists in the definition of an auxiliary funicular structure and the construction of the corresponding force polygons, thus generating two reciprocal diagrams, namely the form and the force diagrams. This approach is based on the early work on graphical force equilibrium by Stevin [12], later developed by Varignon [13] who introduced the relationship between the funicular polygon and the polygon of forces (Figure 1) and eventually formalized within the field of projective geometry by Culmann [3] and Cremona [2].

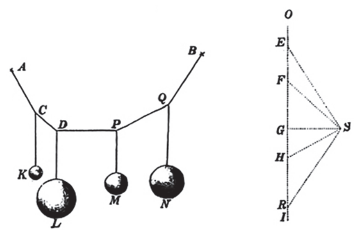


Figure 1: Funicular structure and force polygons by Varignon [13]

## 2. Evaluation of Global Equilibrium in 3D using Graphic Statics

Two procedures have been recently described that address the problem of global equilibrium in three dimensions using graphic statics and lead to the definition of vector-based (Schrems and Kotnik [11]) and polyhedral-based (Akbarzadeh et al. [1]) form and force diagrams respectively. A third vector-based procedure, which is reviewed in details in this paper, is based on the use of graphic statics and projections.

### 2.1 Global Equilibrium through Iterations

The procedure described by Schrems and Kotnik [11] for the evaluation of the global equilibrium in three-dimensions is based on an iterative process. In the first iteration (Figure 2), chosen three arbitrary skew forces $\{\mathbf{F_{1j}}, \mathbf{F_{2j}}, \mathbf{F_{3j}}\}$ from the input system, the magnitudes and lines of action of a pair of forces $\{\mathbf{R_{1j}}, \mathbf{R_{2j}}\}$ are determined so that they are equivalent to the chosen three forces. The procedure to compose the three forces into two is similar to one described by Saviotti ([10], p. 54) and relies on the construction of a specific auxiliary structure. The method is grounded on the use of 3D vector-based form F and force F* diagrams. The pair of forces $\{\mathbf{R_{1j}}, \mathbf{R_{2j}}\}$ here obtained represents one of the infinite pairs of forces that are all equivalent to the force-couple system $\{\mathbf{r_j}, \mathbf{m_{Oj}}\}$ of $\{\mathbf{F_{1j}}, \mathbf{F_{2j}}, \mathbf{F_{3j}}\}$ with respect to an arbitrary point $O$. After the first iteration, another force is chosen from the input system and processed together with the pair of forces found in the previous iteration. By reducing at each iteration the number of the forces of the input system by one, the procedure is repeated until one global pair of forces $\{\mathbf{R_1}, \mathbf{R_2}\}$ is found. This one represents one of the infinite possible pairs of forces that are all equivalent to the force-couple system $\{\mathbf{r}, \mathbf{m_O}\}$ of the input system of forces. Based on $\{\mathbf{R_1}, \mathbf{R_2}\}$, in case the system is externally statically determinate, in a second step the reaction forces at the supports are obtained. This is achieved using an auxiliary tetrahedral structure and superimposing the reaction forces generated on the supports by each force of the pair.

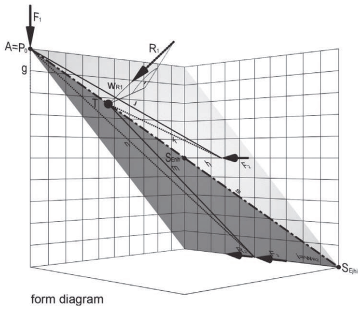


Figure 2: Vector-based 3D global equilibrium by Schrems and Kotnik [11]

### 2.2 Global Equilibrium using an Equilibrium Plane

The method explained by Akbarzadeh et al. [1] is comparable to a procedure described by Saviotti ([10], p. 57) to find the resultant force and the resultant couple of a given system of forces in three dimensions. An equilibrium plane perpendicular to the free resultant **r*** of the input forces in the force diagram is firstly defined and the intersection points of the lines of action of the input forces with the plane are found. The line of action of the resultant force **r** passes through the centroid of the previously defined intersection points, weighted by the magnitudes of the components of the input forces perpendicular to the equilibrium plane. If not null, the components of the input forces parallel to the equilibrium plane produce a resultant couple **m** that can be represented by a free vector parallel to **r**, whose magnitude is invariant to the choice of the reference point *O*. The resultant force-couple system $\{\mathbf{r}, \mathbf{m}\}$ is thus equivalent to the input system of forces. Only if the input system do not produce a resultant couple (i.e. the lines of action of all the input forces intersect in one point in space or the forces are all parallel) the input forces are equivalent to a resultant force **r** alone, with its associated line of action. In this case, and provided the system is externally statically determinate, the resultant force **r** is then used to build 3D polyhedral-based form and force diagrams (Figure 3) to assess the reaction forces at the supports.

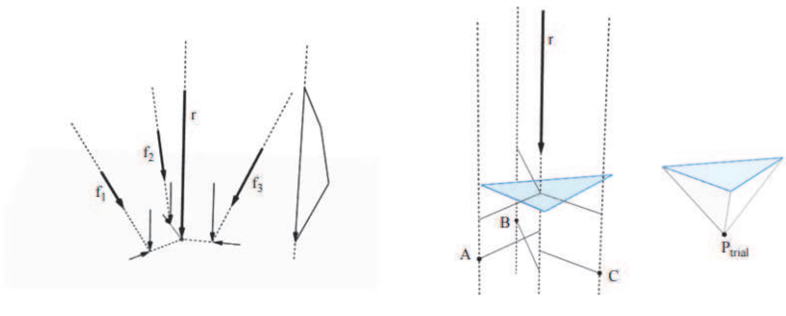


Figure 3: Polyhedral-based 3D global equilibrium by Akbarzadeh et al. [1]

### 2.3 Global Equilibrium based on Projections

The method that uses graphic statics and projections for the evaluation of global equilibrium has been originally introduced by Culmann ([3], pp. 141-144) to determine the centre of a system of parallel forces in space. As explained by Mayor ([6], pp. 103-109), a similar procedure was used by Cremona to find the equilibrium of concurrent forces in space and eventually Saviotti ([10], pp. 57-58) described an equivalent process to compose arbitrary forces in space. Based on this method, a given system of *n* forces $\Sigma$ can be represented by three skew resultants $\{\mathbf{r_1}, \mathbf{r_2}, \mathbf{r_3}\}$, which are equivalent to the resultant force-couple system $\{\mathbf{r}, \mathbf{m_O}\}$ of $\Sigma$ and which are parallel to three initially freely chosen unit vectors $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$. Moreover, the three resultants $\{\mathbf{r_1}, \mathbf{r_2}, \mathbf{r_3}\}$ can be further composed into a pair of forces or into one force and one couple, showing the equivalence of this method to the other two described in Section 2.1 and 2.2. In the following, the individual steps of the process will be exemplified (Figure 4, left) with a given system $\Sigma$ of five arbitrary skew forces $\{\mathbf{f_1}, \mathbf{f_2}, \mathbf{f_3}, \mathbf{f_4}, \mathbf{f_5}\}$.

*2.3.1 Determination of Three Skew Resultants Equivalent to a Given System of Forces*

In the first step, three unit vectors $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ not lying on the same plane are defined; the line of actions of the forces of the sought triplet $\{\mathbf{r_1}, \mathbf{r_2}, \mathbf{r_3}\}$ will be parallel to $\mathbf{e_1}$, $\mathbf{e_2}$ and $\mathbf{e_3}$ respectively. Based on this set-up, every force $\mathbf{f_i}$ of $\Sigma$ is decomposed into its components $\{\mathbf{f_{i1}}, \mathbf{f_{i2}}, \mathbf{f_{i3}}\}$ along the chosen directions:

$$\mathbf{f_i} = \mathbf{f_{i1}} + \mathbf{f_{i2}} + \mathbf{f_{i3}} = f_{i1}\mathbf{e_1} + f_{i2}\mathbf{e_2} + f_{i3}\mathbf{e_3}$$

where the absolute values of $f_{i1}$, $f_{i2}$ and $f_{i3}$ are the magnitudes of the components $\mathbf{f_{i1}}$, $\mathbf{f_{i2}}$ and $\mathbf{f_{i3}}$ along $\mathbf{e_1}$, $\mathbf{e_2}$ and $\mathbf{e_3}$ respectively. In this way, the initial system of forces $\Sigma$ is replaced by three sets of forces $\Sigma_1$, $\Sigma_2$ and $\Sigma_3$, which are parallel to $\mathbf{e_1}$, $\mathbf{e_2}$ and $\mathbf{e_3}$ respectively. To easily illustrate the process, in the example the unit vectors $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ are chosen to be orthonormal (Figure 4, right).

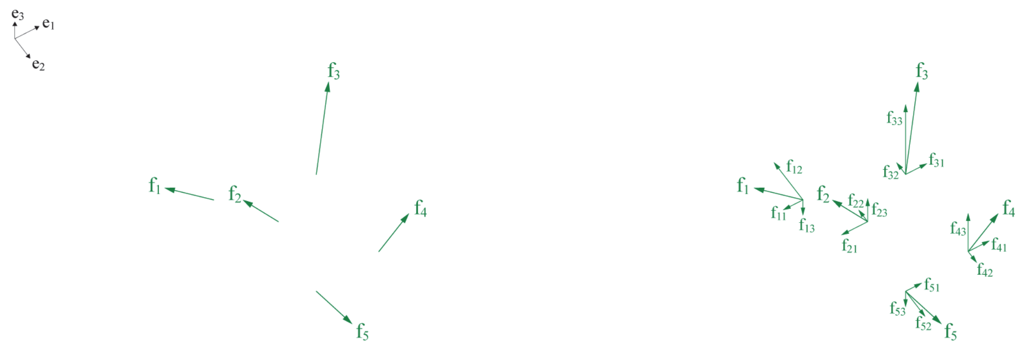


Figure 4: Given system of 5 forces $\Sigma$ (left); decomposition of the forces along $\mathbf{e_1}$, $\mathbf{e_2}$ and $\mathbf{e_3}$ (right)

In the second step, three auxiliary projection planes $\Pi'$, $\Pi''$ and $\Pi'''$ are set to be parallel to the vector pairs $\{\mathbf{e_1}, \mathbf{e_2}\}$, $\{\mathbf{e_1}, \mathbf{e_3}\}$ and $\{\mathbf{e_2}, \mathbf{e_3}\}$ respectively. Moreover, three unit vectors $\{\mathbf{t'}, \mathbf{t''}, \mathbf{t'''}\}$ are introduced, which define the directions of projection onto $\Pi'$, $\Pi''$ and $\Pi'''$; each vector of the triplet $\{\mathbf{t'}, \mathbf{t''}, \mathbf{t'''}\}$ can be freely chosen, provided that it is not parallel to its corresponding projection plane. For convenience of illustration, in the example the vectors $\{\mathbf{t'}, \mathbf{t''}, \mathbf{t'''}\}$ are chosen to be parallel to $\{-\mathbf{e_3}, -\mathbf{e_2}, \mathbf{e_1}\}$ respectively.

The forces $\mathbf{f_{i3}}$ included in the previously defined set $\Sigma_3$, which are all parallel to $\mathbf{e_3}$, are then projected onto $\Pi'''$ through a parallel projection in the direction of $\mathbf{t'''}$, thus generating a new set $\Sigma_3'''$ of coplanar forces $\mathbf{f_{i3}}'''$. In order to find the magnitude and line of action of the resultant $\mathbf{r_3}'''$ of $\Sigma_3'''$, the common procedure for the evaluation of the global equilibrium in 2D graphic statics is here followed. Hence, on the plane $\Pi'''$ an auxiliary funicular structure is built and the corresponding force polygons are constructed, thus generating the form $F_3'''$ and the force $F_3{}^{*}{}'''$ diagrams (Figure 5, left). The magnitude of $\mathbf{r_3}'''$ of $\Sigma_3'''$ is then evaluated. The position of its line of action is found so that the moment generated by $\mathbf{r_3}'''$, with respect to any arbitrary reference point in $\Pi'''$, is equivalent to the resultant couple of $\Sigma_3'''$. Likewise, the forces $\mathbf{f_{i3}}$ included in $\Sigma_3$ are projected onto $\Pi''$ in the direction of $\mathbf{t''}$ to generate $\Sigma_3''$; by construction, the magnitude of the resultant $\mathbf{r_3}''$ of $\Sigma_3''$ is equal to the one of $\mathbf{r_3}'''$ and its line of action is assessed following the previously described 2D graphic statics procedure (Figure 5, right). The forces $\mathbf{r_3}'''$ and $\mathbf{r_3}''$ represent the projections on $\Pi'''$ and $\Pi''$ respectively of the resultant $\mathbf{r_3}$ of $\Sigma_3$. The magnitude of $\mathbf{r_3}$ is therefore the same as $\mathbf{r_3}'''$ and $\mathbf{r_3}''$. The line of action of $\mathbf{r_3}$ is the intersection between the plane containing $\mathbf{r_3}'''$ and parallel to the vector pair $\{\mathbf{t'''}, \mathbf{e_3}\}$ and the plane containing $\mathbf{r_3}''$ and parallel to the vector pair $\{\mathbf{t''}, \mathbf{e_3}\}$ (Figure 7, left). The moment generated by $\mathbf{r_3}$ with respect to any arbitrary reference point O in space is equivalent to the resultant couple of $\Sigma_3$.

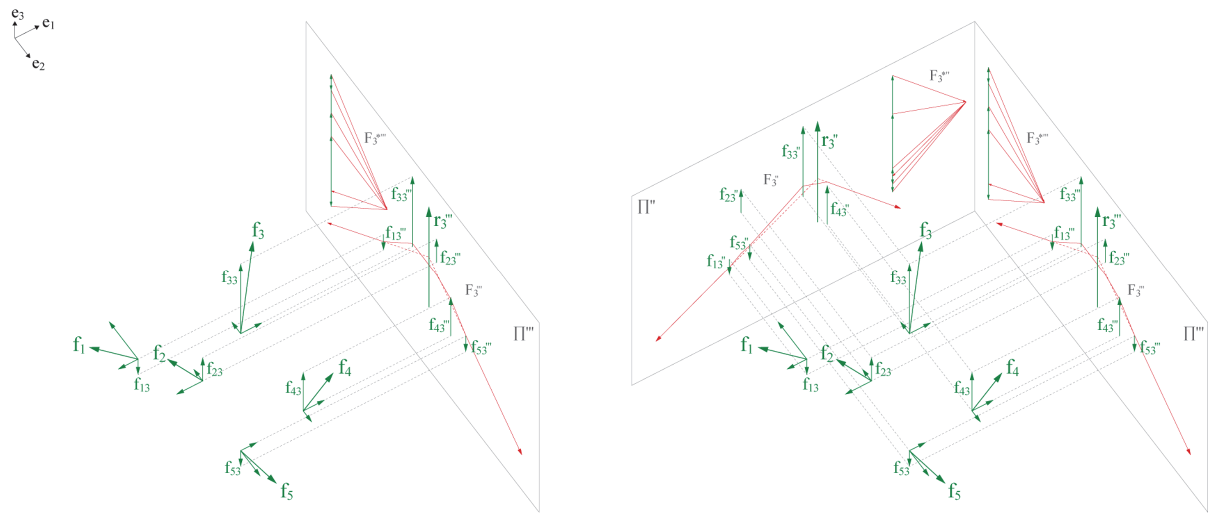


Figure 5: Determination of $\mathbf{r_3}'''$ on Π''' (left); determination of $\mathbf{r_3}'''$ on Π''' and $\mathbf{r_3}''$ on Π'' (right)

The overall process is then repeated to determine the magnitudes and the lines of action of the resultant forces $\mathbf{r_2}$ of $\Sigma_2$ and $\mathbf{r_1}$ of $\Sigma_1$. In the case of $\mathbf{r_2}$, the forces $\mathbf{f_{i2}}$ of $\Sigma_2$ are projected onto Π''' in the direction of $\mathbf{t}'''$ and onto Π' in the direction of $\mathbf{t}'$ (Figure 6, left); once the resultant forces $\mathbf{r_2}'''$ of $\Sigma_2'''$ and $\mathbf{r_2}'$ of $\Sigma_2'$ are found, the line of action of $\mathbf{r_2}$ is found at the intersection of the plane containing $\mathbf{r_2}'''$ and parallel to the vector pair $\{\mathbf{t}''', \mathbf{e_2}\}$ and the plane containing $\mathbf{r_2}'$ and parallel to the vector pair $\{\mathbf{t}', \mathbf{e_2}\}$ (Figure 7, left). In the case of $\mathbf{r_1}$, the forces $\mathbf{f_{i1}}$ of $\Sigma_1$ are projected onto Π'' along $\mathbf{t}''$ and onto Π' along $\mathbf{t}'$ (Figure 6, right); based on the resultant forces $\mathbf{r_1}''$ of $\Sigma_1''$ and $\mathbf{r_1}'$ of $\Sigma_1'$, the line of action of $\mathbf{r_1}$ is represented by the intersection of the plane containing $\mathbf{r_1}''$ and parallel to the vector pair $\{\mathbf{t}'', \mathbf{e_1}\}$ and the plane containing $\mathbf{r_1}'$ and parallel to the vector pair $\{\mathbf{t}', \mathbf{e_1}\}$ (Figure 7, left).

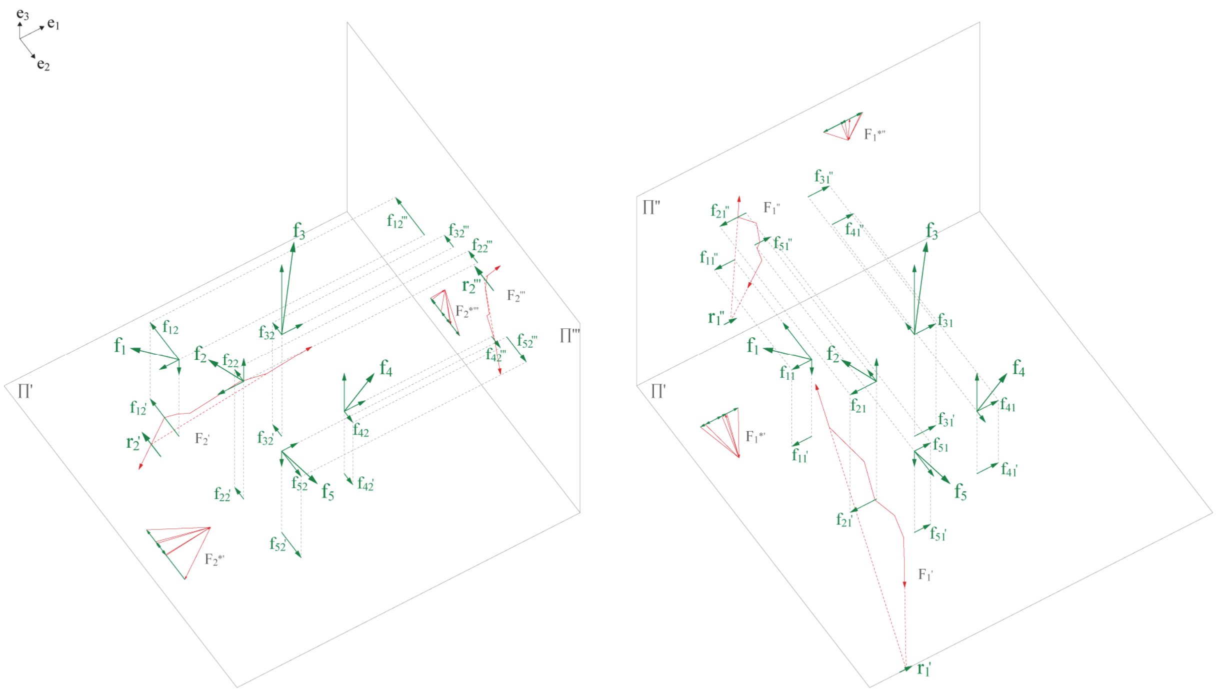


Figure 6: Determination of $\mathbf{r_2}'''$ on Π''' and $\mathbf{r_2}'$ on Π' (left); determination of $\mathbf{r_1}''$ on Π'' and $\mathbf{r_1}'$ on Π' (right)

As a result, the magnitudes of $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ and their lines of action are determined to be equivalent to the resultant force-couple systems of $\Sigma_1$, $\Sigma_2$ and $\Sigma_3$ respectively. Moreover, considering that $\Sigma_1$, $\Sigma_2$ and $\Sigma_3$ are themselves equivalent to the given system of forces $\Sigma$, the triplet of forces $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ is also equivalent to $\Sigma$ (Figure 7, right). In fact, the triplet of forces here obtained represents one of the infinite triplets of forces that are all equivalent to the force-couple system $\{\mathbf{r}, \mathbf{m}_\mathbf{O}\}$ of $\Sigma$. By varying the unit vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ other triplets can be found. Depending on the specific problem to be solved, one or more resultants of the triplet $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ might be null. In this particular case, the system might also produce a couple $\mathbf{m}$, whose magnitude is invariant to the choice of the reference point $O$.

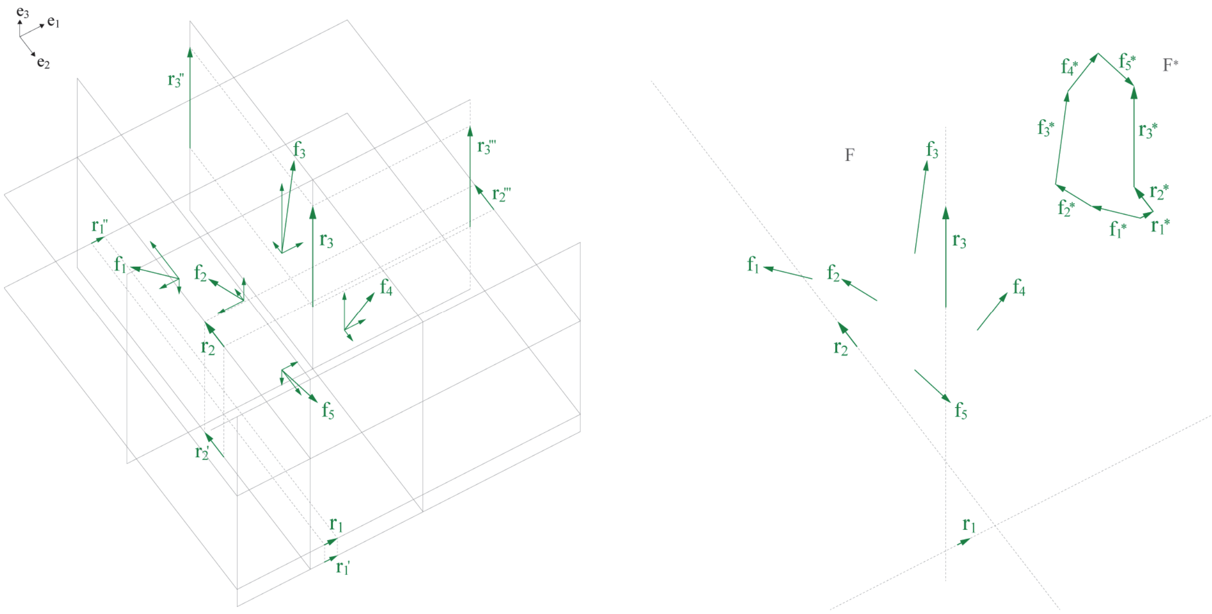


Figure 7: Determination of $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ (left); the given system of forces $\Sigma$ with the resultants $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ (right)

### *2.3.2 Composition of Three Skew Resultants into Two Resultants*

The triplet of skew resultant forces $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ defined in the previous section can be further reduced to an equivalent pair of resultants $\{\mathbf{r}_a, \mathbf{r}_b\}$, using for example the procedure described by Saviotti [10] and Schrems and Kotnik [11].

An alternative method is here described based on the use of a simple 3D funicular structure (Figure 8). At first, after selecting one of the forces of the triplet (for example $\mathbf{r}_3$) in the 3D form diagram F, a point $C$ on its line of action is freely chosen. The point $C$ thus defines together with the lines of action of the other two forces of the triplet ($\mathbf{r}_1$ and $\mathbf{r}_2$ in the example) two planes $\Pi_a$ and $\Pi_b$ respectively. A plane $\Pi_c$ is then freely chosen that contains the line of action of the selected force of the triplet ($\mathbf{r}_3$ in the example) and that intersects the lines of action of the other two forces of the triplet ($\mathbf{r}_1$ and $\mathbf{r}_2$ in the example) in two points $A$ and $B$. As a result, the two segments $AC$ and $CB$, which belong to the sought 3D funicular structure, are obtained. Based on this setup, the 3D force diagram F* can be built; here the forces $\mathbf{f}_{ac}$* and $\mathbf{f}_{cb}{}^*$, which correspond to the segments $AC$ and $CB$ respectively, meet in P*. From the point P*, it is possible to construct the forces $-\mathbf{r}_a$* and $-\mathbf{r}_b$* that close the 3D skew force polygon $\mathbf{r}_2^*|\mathbf{r}_3^*|\mathbf{r}_1^*|-\mathbf{r}_a^*|-\mathbf{r}_b^*$. In the 3D form diagram F, the lines of actions of the forces $-\mathbf{r}_a$ and $-\mathbf{r}_b$ are parallel to $-\mathbf{r}_a$* and $-\mathbf{r}_b$* and pass through the points $A$ and $B$ respectively. By construction, after inverting the forces $-\mathbf{r}_a$ and $-\mathbf{r}_b$, a pair of resultants $\{\mathbf{r}_a, \mathbf{r}_b\}$ is determined. This represents one of the infinite pairs of forces that are all equivalent to the force-couple system $\{\mathbf{r}, \mathbf{m}_\mathbf{O}\}$ of $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ and as such of $\Sigma$ with respect to an arbitrary reference point $O$.

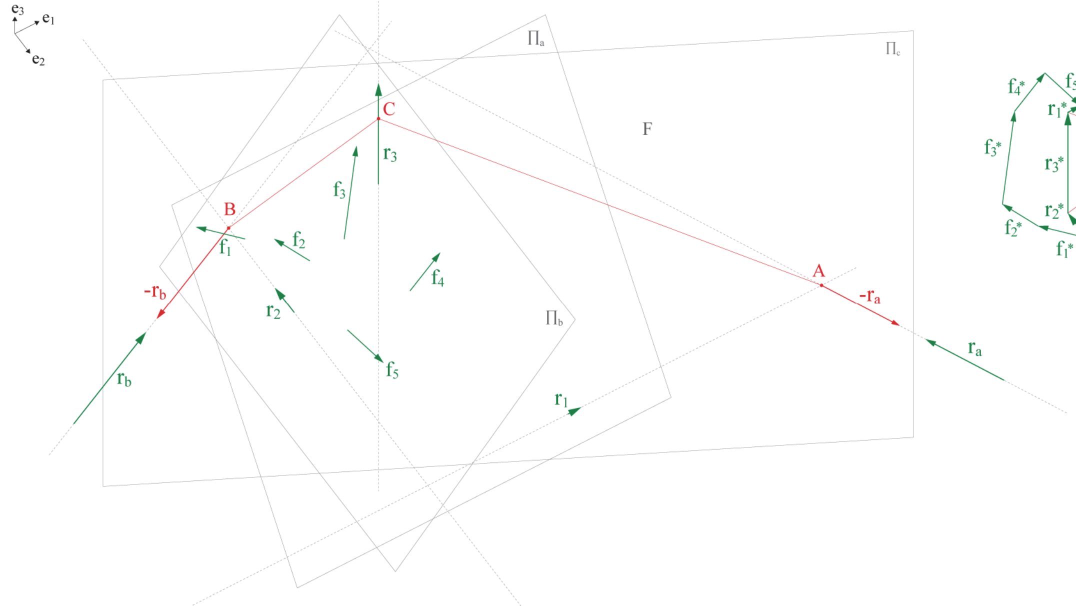


Figure 8: Composition of three skew resultants $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ into two resultants $\{\mathbf{r}_a, \mathbf{r}_b\}$

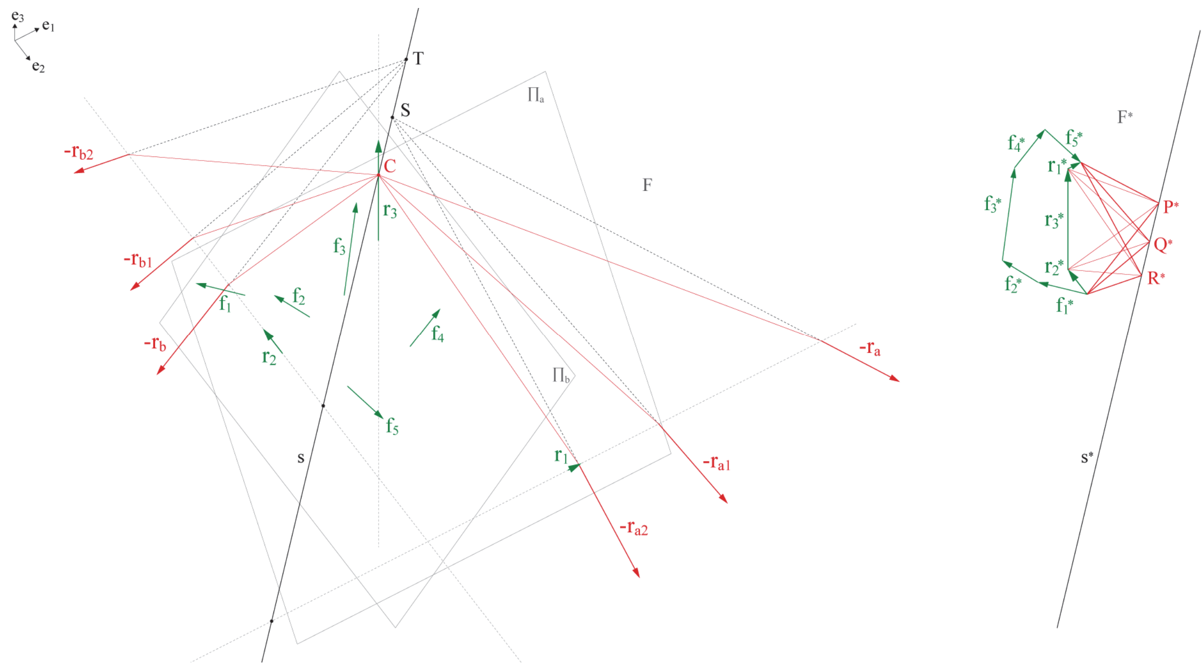


Figure 9: Construction of pairs of conjugate forces belonging to the planes $\Pi_a$ and $\Pi_b$

In fact, the lines of action of the forces $\mathbf{r_a}$ and $\mathbf{r_b}$ are conjugate (or reciprocal) lines and $\mathbf{r_a}$ and $\mathbf{r_b}$ are conjugate (or reciprocal) forces; that is, according to the input system of forces $\Sigma$, if the line of action of one of the forces is fixed, the line of action of the other is univocally determined. This geometric property was firstly demonstrated by Möbius [7] and later further elaborated by Cremona [2] within the framework of projective geometry. By rotating the plane $\Pi_c$ around the line of action of the selected force of the triplet ($\mathbf{r_3}$ in the example), infinite pairs of conjugate forces $\{\mathbf{r_{ai}}, \mathbf{r_{bi}}\}$ that belong to the planes $\Pi_a$ and $\Pi_b$ respectively can be found (Figure 9). The lines of action of all the forces $\mathbf{r_{ai}}$ on $\Pi_a$ meet in a point *S* and the ones of the forces $\mathbf{r}_{bi}$ on $\Pi_b$ meet in a point *T*. The points *S*, *T* and *C* belong to the same line *s*, which coincides with the intersection of the planes $\Pi_a$ and $\Pi_b$; by construction, *s* also intersects the lines of action of $\mathbf{r}_1$ and $\mathbf{r}_2$. It is possible to take advantage of this property to easily find specific pairs of resultants $\{\mathbf{r}_a, \mathbf{r}_b\}$, one of which is passing through a given point or it is parallel to a specific line. Moreover, by moving the point *C* along the line of action of the selected force of the triplet, the planes $\Pi_a$ and $\Pi_b$ vary and other families of pairs of conjugate forces can be obtained.

### *2.3.3. Determination of One Resultant and One Couple*

The method explicated in Section 2.3.1 to determine three skew resultants equivalent to a given system of forces $\Sigma$ through projections can be used to find directly a resultant force-couple system $\{\mathbf{r}, \mathbf{m}\}$ of $\Sigma$. In fact, if the unit vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are chosen to be orthonormal and $\mathbf{e}_3$ is set to be parallel to the free resultant $\mathbf{r}^*$ of $\Sigma$ in the force diagram F*, the method of the projections yields the position of the line of action of the resultant $\mathbf{r}$, which coincides with the resultant $\mathbf{r}_3$ of the set of forces $\Sigma_3$ parallel to $\mathbf{e}_3$. Furthermore, the set of forces $\Sigma_1$ and $\Sigma_2$, which are parallel to $\mathbf{e}_1$ and $\mathbf{e}_2$ respectively, if not empty, generate a resultant couple $\mathbf{m}$. This can be represented by a free vector with two components, $\mathbf{m}_{||}$ parallel to $\mathbf{r}$ and $\mathbf{m}_\perp$ perpendicular to $\mathbf{r}$.

Likewise, and in case the forces of the given system $\Sigma$ are decomposed at the intersection points of their lines of action with the projection plane $\Pi'$ parallel to the vector pair $\{\mathbf{e}_1, \mathbf{e}_2\}$ and perpendicular to $\mathbf{e}_3$, the method of the projections is totally equivalent to the procedure described by Akbarzadeh et al. [1]. The line of action of the resultant $\mathbf{r}$, which coincides with the resultant $\mathbf{r}_3$ parallel to $\mathbf{e}_3$, represents the central axis *a* of the system $\Sigma$ and the plane $\Pi'$ is the orthographic plane (Cremona [2]). If not null, the components of the input forces $\Sigma$ parallel to $\mathbf{e}_1$ and $\mathbf{e}_2$ generate a resultant couple $\mathbf{m}$ that can be represented by a free vector parallel to $\mathbf{r}$ and perpendicular to $\Pi'$; hence, the component $\mathbf{m}_\perp$ of $\mathbf{m}$ is null. Furthermore, if the resultant couple $\mathbf{m}$ is represented in the form diagram by a couple of forces $\{\mathbf{f}, \mathbf{-f}\}$ on $\Pi'$ and the line of action of $\mathbf{f}$ is made to intersect the central axis, $\mathbf{f}$ and $\mathbf{r}$ can be composed into a force $\mathbf{f}_r$. This one and the force $\mathbf{-f}$ are conjugate as described in Section 2.3.2 and the system $\Sigma$ is thus equivalent to the pair of forces $\{\mathbf{f}_r, \mathbf{-f}\}$ (Möbius [7]). Based on this construction it is immediate to show that the projections of the lines of action of conjugate forces on the orthographic plane are parallel lines (Cremona [2]).

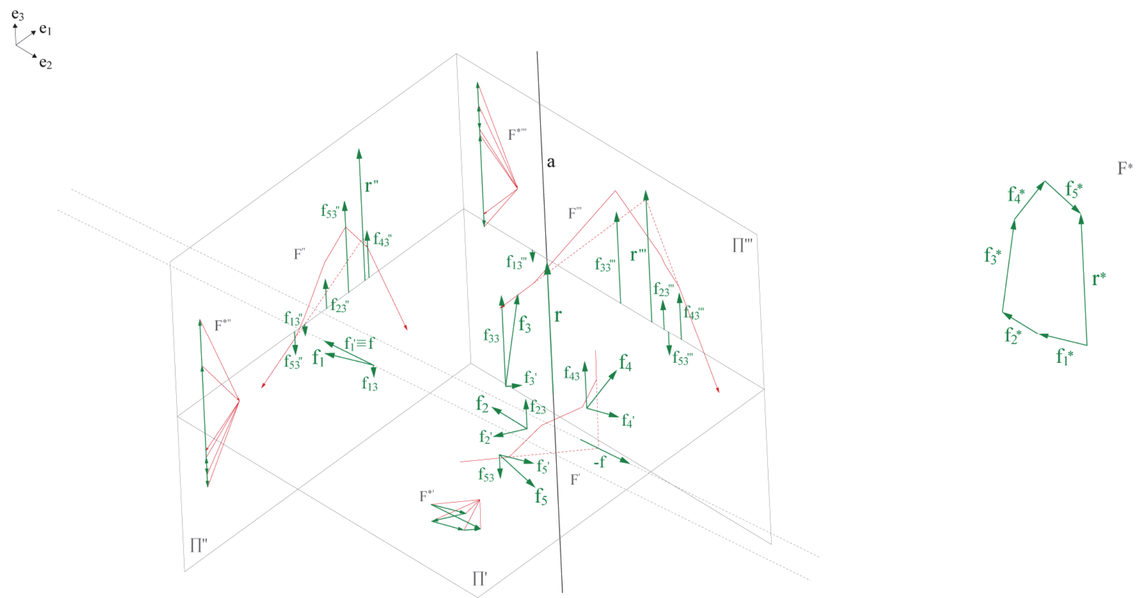


Figure 10: Determination of one resultant and one couple of $\Sigma$

### *2.3.4. Evaluation of Reactions at the Supports*

Once the triplet of forces $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ equivalent to the given system of forces $\Sigma$ is found following the method described in Section 2.3.1, the reactions at the supports can be evaluated according to the specified boundary conditions. Given three orthonormal unit vectors $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$, the boundary condition can be defined by setting a certain number of support conditions in the form of restrains along $\mathbf{s}_1$, $\mathbf{s}_2$ and $\mathbf{s}_3$. It is here important to distinguish the cases of externally statically determinate and indeterminate systems. To easily illustrate the process, in the example the unit vectors $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ are chosen to coincide with the unit vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ previously introduced.

In the case of an externally statically determinate system in space, six support conditions have to be set (Pirard [9], p. 309). A simple configuration is given, for example, by three support points $S_A$, $S_B$ and $S_C$, where $S_A$ is restrained along $\mathbf{s}_3$, $S_B$ is restrained along $\mathbf{s}_3$ and $\mathbf{s}_2$ and $S_C$ is restrained along $\mathbf{s}_3$, $\mathbf{s}_2$ and $\mathbf{s}_1$. In analogy with 2D graphic statics, a simple auxiliary structure can be used to evaluate the reaction forces $\mathbf{r}_{sA}$, $\mathbf{r}_{sB}$ and $\mathbf{r}_{sC}$ produced by the forces $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ at the support points $S_A$, $S_B$ and $S_C$ respectively. As such, an internally statically determinate 3D truss made of three tetrahedra (6 nodes, 12 edges) is constructed, whose nodes are the three support points $S_A$, $S_B$ and $S_C$ and other three freely chosen points on the lines of action of $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ respectively (Figure 11); in case one or more forces of the triplet $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ are null, the corresponding nodes of the truss are not generated. The equilibrium of the forces is then imposed node by node in the force diagram F*, ranking the nodes at every step of the process in ascending order according to the number of unknown forces (Jasienski et al. [4]).

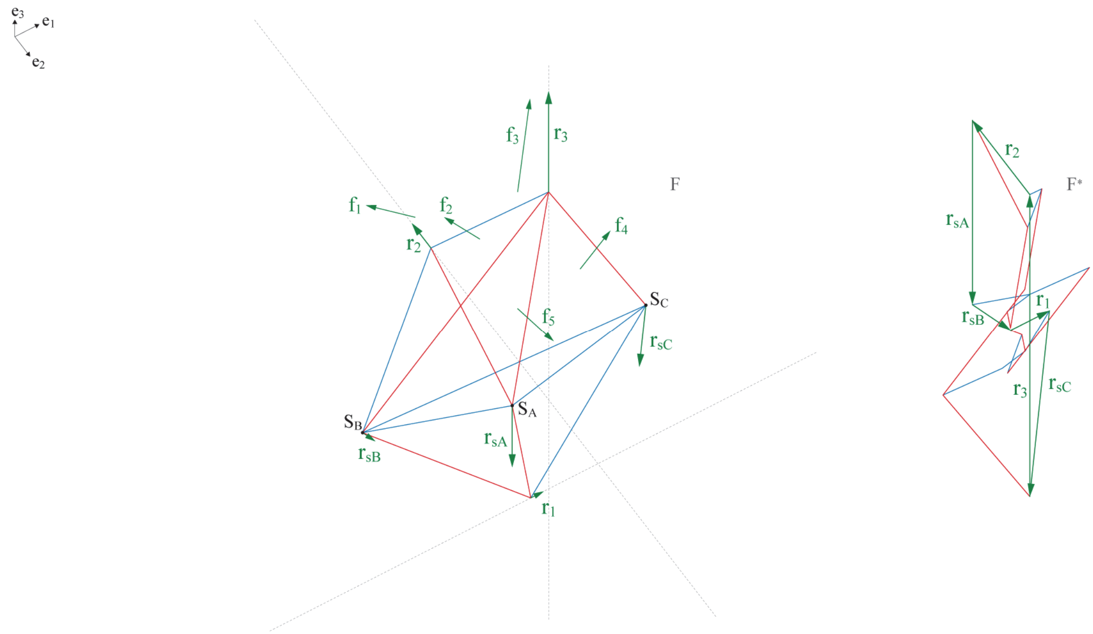


Figure 11: Evaluation of the reactions $\mathbf{r}_{sA}$, $\mathbf{r}_{sB}$ and $\mathbf{r}_{sC}$ at the supports (red-tension, blue-compression)

In the case of an externally statically indeterminate system, more than six support conditions are defined. This is the situation, for example, of a system with more than three support points. Considering the global equilibrium alone, according to the Lower Bound theorem of the Theory of Plasticity (Muttoni et al. [8]) infinite solutions to the problem are possible. In order to explore the solution space, those support conditions other than the six necessary ones to attain the determinacy of the system have to be regarded as parameters in the problem. That is, every extra reaction force is given a trial value and processed together with the forces included in the given system $\Sigma$ following the method explained in Section 2.3.1. By varying the values assigned to the extra reaction forces, the solution space can be explored parametrically.

In the force diagram F*, the forces $\mathbf{f_i}$ of Σ and the reactions $\{\mathbf{r_{sA}}, \mathbf{r_{sB}}, \mathbf{r_{sC}}\}$ define a closed 3D skew force polygon. If a structure is given on which the forces $\mathbf{f_i}$ of Σ and $\{\mathbf{r_{sA}}, \mathbf{r_{sB}}, \mathbf{r_{sC}}\}$ are applied, the closed 3D skew force polygon can be used as a convenient base to construct the 3D force diagram of the structure itself, as described in Jasienski et al. [4].

## 3. Conclusion

The evaluation of the equilibrium of a system of forces that fulfils specified boundary conditions has been addressed in the present paper within the framework of graphic statics. Three main procedures to solve this problem in three dimensions using graphic statics have been reviewed and particular emphasis has been put on a vector-based graphical method to evaluate the global equilibrium by means of projections. It has been shown how this method, which leads to the definition of three skew resultants parallel to three initially chosen unit vectors, is equivalent to the other two methods. Eventually, given the three resultants it has been described how to evaluate the reactions at the supports according to predefined boundary conditions both in the case of externally statically determinate and indeterminate systems. Being based on simple geometrical operations only for the construction of 2D and 3D funicular structures, the method of the projections shows a great potential to be implemented into a digital tool within a 3D software environment. This aspect will be addressed in further developments of the research.

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